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Properties of definite Integrals

- $$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$$

follows from the sum rule for limits

pf

$$\int_a^b (f+g)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} (f+g)(c_k)\,\Delta x_k$$

↑ definition

width of the k-th subinterval (Δx_k)

c_k: point in the k-th subinterval of $[a,b]$ (divided into n equal sub intervals)

sum rule for limits ↓

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k + \sum_{k=1}^{n} g(c_k)\,\Delta x_k \right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k + \lim_{n\to\infty} \sum_{k=1}^{\infty} g(c_k)\,\Delta x_k$$

↓ definition

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

□

- $$\int_a^b (k f(x))\,dx = k \int_a^b f(x)\,dx \qquad k \text{ is a constant}$$

using the rule $\lim_{n\to\infty} (k\,h(n)) = k \lim_{n\to\infty} h(n)$.

Definition 1.3. Let $f(x)$ be continuous on $[a, b]$. Consider the partition: $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$. For any $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k$ is a fixed number, called definite (Riemann) integral of $f(x)$ on $[a, b]$, denoted by $\int_a^b f(x)\,dx$, i.e.,

$$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx}$$

Hard Theorem: Let f be a piecewise continuous function, then $\int_a^b f(x)\,dx$ is well-defined. I.e. The limit in the preceding definition exists, and is independent of the choices of c_k.

Remark. The "Lebesque integral" is well-defined for more general functions.

Example 1.3. Evaluate $\int_2^3 x\,dx$ using the left Riemann sum with n equally spaced subintervals.

Fundamental thm of Calculus:

An antiderivative of x is $F(x) = \frac{x^2}{2}$

$$\int_2^3 x\,dx = \left.\frac{x^2}{2}\right|_2^3 = \frac{3^2}{2} - \frac{2^2}{2} = \frac{5}{2} \quad \Box$$

Example 1.4. Evaluate $\int_0^1 x^2\,dx$ using the right Riemann sum with n equally spaced subintervals.

Solution. Let $f(x) = x^2$. Consider the partition of $[0,1]$: $x_k = \frac{k}{n}$, $k = 0, \ldots, n$.
Right Riemann sum: on $[x_{k-1}, x_k]$, $c_k = x_k = \dfrac{k}{n}$.

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{(n+1)(2n+1)}{6n^2}.$$

$$\left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

$$\text{So, } \int_0^1 x^2\,dx = \lim_{n\to+\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}. \qquad \blacksquare$$

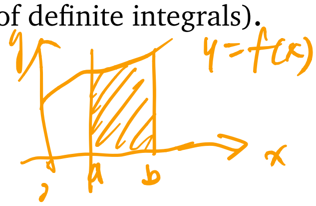
Fundamental thm of Calculus: Anti-derivative of x^2 $= \dfrac{x^3}{3}$

$\int_0^1 x^2\,dx = \dfrac{x^3}{3}\Big|_0^1 = \dfrac{1^3}{3} - \dfrac{0}{3} = \dfrac{1}{3}$ $\square$

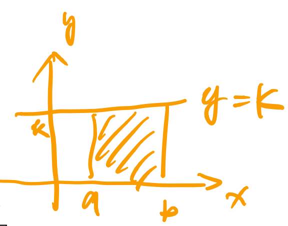
Remark. It's so complicated to used definition to compute $\int_a^b f(x)\,dx$. Later, we will discuss another easier method: fundamental theorem of calculus.

Using the interpretation of definite integrals as signed areas and its definition as limits of Riemann sums, we have:

Theorem 1.1 (Properties of definite integrals)**.**



1. $\boxed{\int_a^a f(x)\,dx = 0}$



2. $\boxed{\int_a^b k\,dx = \underset{\uparrow}{k}\underbrace{(b-a)}}$ → width of the rectangle
   height of rectangle

3. $\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

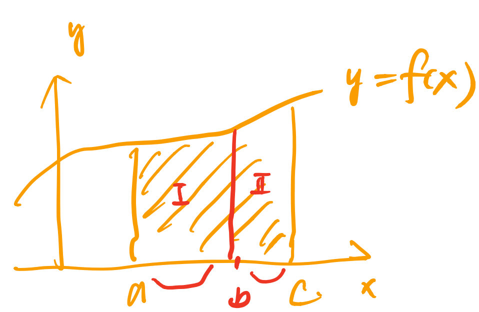
   $\boxed{kf(x)\,dx = k\int_a^b f(x)\,dx}$

4. if $a < b$,

   convention

   $\boxed{\int_b^a f(x)\,dx \triangleq -\int_a^b f(x)\,dx}$ ($\triangleq$, defined to be) [a, b]

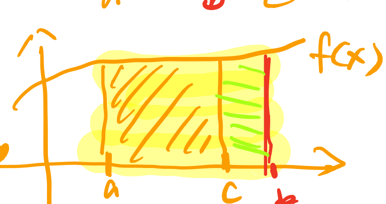
   area of region I



5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$ ← area of region II

area of the shaded region
= area of yellow region − area of green region

$\int^b f(x)\,dx = \int^b f(x)\,dx$



$= \int_a f(x)\,dx - \int_c f(x)\,dx$

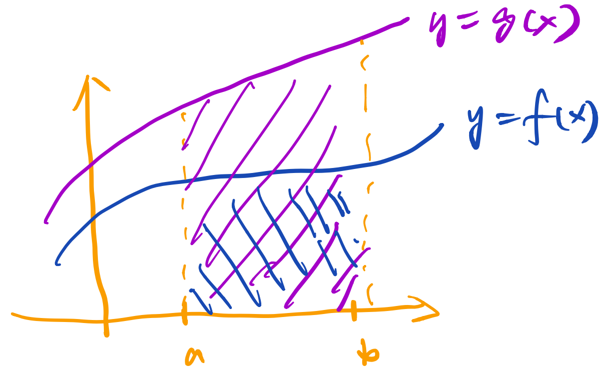
$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

6. *if* $f(x) \le g(x)$ *on* $[a, b]$, *then*

$$\boxed{\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx}$$

area of the blue region.

area of purple region



2 The fundamental Theorem of Calculus

Notation:

$\displaystyle\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$: definite integral of function f on $[a, b]$, which is a number.

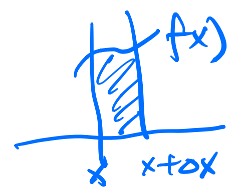
$\displaystyle\int_a^x f(t)\,dt$: definite integral of function f on $[a, x]$, it can be regarded as a function of x.

Theorem 2.1 (Fundamental Theorem of Calculus)**.**

Assume $f(x)$ is continuous.

function in x

1. $\boxed{\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)}$ *(i.e. $\displaystyle\int_a^x f(t)\,dt$ is an anti-derivative of $f(x)$)*



2. *Let $F(x)$ be any anti-derivative of $f(x)$, $F'(x) = f(x)$, then*

$$\boxed{\int_a^b f(x)\,dx = F(x)\big|_a^b := F(b) - F(a)}.$$

when $\Delta x \to 0$ $f(x)\Delta x \approx \displaystyle\int_x^{x+\Delta x} f(t)\,dt$

Heuristic explanation:

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt}{\Delta x} = \lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x)\Delta x}{\Delta x} = f(x)$$

Example 2.1. Computing the integrals in Examples 1.3 and 1.4 using the fundamental theorem of calculus.

Example 2.2. Derive Theorem 1.1 from the corresponding theorem for indefinite integrals and the fundamental theorem of calculus.

Remark.

1. Differentiation $\xleftrightarrow{\text{Fundamental thm of calculus}}$ Integration

   $F'(x) = f(x)$ $\qquad\qquad$ $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

2. Anti-derivative $F(x)$ is not unique. Which one should we choose?

   Another anti-derivative: $\tilde{F}(x) = F(x) + C$, then

   $$\tilde{F}(b) - \tilde{F}(a) = (F(b) + \cancel{C}) - (F(a) + \cancel{C}) = F(b) - F(a).$$

   so, it does not matter, we can choose any anti-derivative.

Example 2.3.

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\bigg|_1^9 = \frac{2}{3}x^{3/2}\bigg|_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}.$$

Example 2.4. Evaluate $\displaystyle\int_1^2 \ln x\,dx$.

We first find one antididerivative of $\ln x$,

$$\int u v'\,dx = uv - \int v u'\,dx.$$

$$\begin{aligned} \int \underbrace{1}_{v'} \cdot \underbrace{\ln x}_{u}\,dx &= \underbrace{x}_{v} \underbrace{\ln x}_{u} - \int \overbrace{1}^{x \cdot \frac{1}{x}}\,dx \quad \text{(integration by parts)} \\ &= x\ln x - x + C. \end{aligned}$$

$v' = 1 \qquad v = x$

$u' = \frac{1}{x} \qquad u = \ln x$

So, $\displaystyle\int_1^2 \ln x\,dx = (x\ln x - x)|_1^2 = 2\ln 2 - 1.$ $\square$

$(2\ln 2 - 2) - (1 \cdot \ln 1 - 1)$

Example 2.5. Let

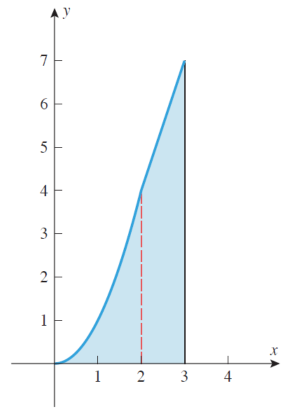
$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}.$$

Find $\displaystyle\int_0^3 f(x)\,dx$.

$$= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

$$= \frac{x^3}{3}\bigg|_0^2 + \left(\frac{3x^2}{2} - 2x\right)\bigg|_2^3$$



$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx \qquad \text{(integrate separately)}$$
$$= \frac{x^3}{3}\Big|_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}.$$

□.

Exercise 2.1.

$u = x^2$. $du = 2x\,dx$

$\int 2x e^{x^2} dx = \int e^u du = e^u + C$
$= e^{x^2} + C$

$e^{x^2}\Big|_0^1 =$ 1. $\displaystyle\int_0^1 2xe^{x^2}\,dx = e - 1.$

2. $\displaystyle\int_{-1}^2 |x|\,dx = \frac{5}{2}.$

$f(x)$

$f(x) = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x \leq 0 \end{cases}$

$\int_{-1}^0 (-x)\,dx + \int_0^2 x\,dx$

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Example 2.6. Compute $\dfrac{d}{dx}$ for (1) $\displaystyle\int_1^x e^{t^2}\,dt$, (2) $\displaystyle\int_{x^2}^{x^3} e^{t^2}\,dt$, (3) $\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt$.

Solution. It's impossible to get explicit formula for $F(t) = \int e^{t^2}\,dt$.

1. By fundamental theorem of calculus (1), we have
$$\frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}.$$

2. Let $F'(t) = e^{t^2}$, then
$$\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = \frac{d}{dx}(F(x^3) - F(x^2)) = F'(x^3)\cdot 3x^2 - F'(x^2)\cdot 2x = e^{x^6}\cdot 3x^2 - e^{x^4}\cdot 2x.$$